Exercises and problems about diagram groups

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Exercise 1. 1. Show that the semigroup presentation $\mathcal{P} = \langle a, b \mid a = b \rangle$ is aspherical, i.e. the diagram group $D(\mathcal{P}, w)$ is trivial for every word $w \in \{x, y\}^*$.

2. Draw the Squier complex $S(\mathcal{P}, a)$ where $\mathcal{P} = \langle a, b, c, d \mid a = b, b = c, c = a, c = d, d = a \rangle$. Deduce that the diagram group $D(\mathcal{P}, a)$ is free of rank two.

3. Draw the Squier complex $S(\mathcal{P}, a^2)$ where $\mathcal{P} = \langle a, b, c \mid a = b, b = c, c = a \rangle$. Deduce that the diagram group $D(\mathcal{P}, a^2)$ is free abelian of rank two.

Exercise 2. Let S be a set. A *rewriting rule* is an ordered pair $\ell \to r$ of elements of S. A *rewriting systems* is a collection of rewriting rules. We write $\overset{*}{\to}$ the transitive-closure of $\to$. A rewriting system is *terminating* if every chain $\ell_1 \to \ell_2 \to \cdots$ eventually terminates. It is *locally confluent* if, for all $\ell, r_1, r_2 \in S$ satisfying $\ell \to r_1, r_2$, there exists $r \in S$ such that $r_1, r_2 \overset{*}{\to} r$.

1. Fix a terminating and locally confluent rewriting system. Let $a, b \in S$ be two elements in the class of the equivalence relation generated by $\to$. Prove that there exists a unique $c \in S$ such that $a, b \overset{*}{\to} c$ and such that there is no element $d \in S$ satisfying $c \to d$.

2. Let $\mathcal{P}$ be semigroup presentation. Consider the rewriting system given by diagrams over $\mathcal{P}$ and dipole reduction. Prove that it is locally confluent and terminating. Deduce that every diagram admits a unique reduction.

Exercise 3. Prove that finitely generated diagram groups have solvable word problems.

Exercise 4. Prove that the diagram group $D(\mathcal{P}, ab)$ where

$$\mathcal{P} = \langle a, b, p, q, r \mid a = ap, b = pb, p = q, q = r, r = p \rangle$$

is isomorphic to $\mathbb{Z} \wr \mathbb{Z} := \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}\right) \rtimes \mathbb{Z}$.

Exercise 5. Let X be a median graph.

1. Let α, β be two geodesics with the same endpoints. Prove that there exists a sequence of geodesics

$$\gamma_0 := \alpha,\ \gamma_1, \gamma_2, \dots, \gamma_{n-1},\ \gamma_n := \beta$$

such that, for every $0 \leq i \leq n-1$, γ_{i+1} is obtained from γ_i by *fliping a 4-cycle* (i.e. replacing two consecutive edges e_1, e_2 with two edges f_1, f_2 whenever e_1, e_2, f_2, f_1 define a 4-cycle).

2. Let α be a path. Prove that there exists a sequence of path with the same endpoints

$$\gamma_0 := \alpha,\ \gamma_1, \gamma_2, \dots, \gamma_{n-1},\ \gamma_n$$

such that γ_n is a geodesic and such that each γ_{i+1} is obtained from γ_i by adding or removing a backtrack or by fliping a 4-cycle.

3. Conclude that, given two arbitrary paths with the same endpoints, one can always be obtained from the other by adding or removing backtracks and by fliping 4-cycles.

Exercise 6. Let G be a group acting on a median graph X. Let $\mathfrak{H}$ denote the graph whose vertices are the G-orbits of hyperplanes in X and whose edges connect two classes whenever they contain transverse hyperplanes. The goal is to construct a morphism from G to the right-angled Coxeter group $C(\mathfrak{H})$.

1. For every oriented path γ in X, let $w(\gamma)$ denote the word written over the vertex-set $V(\mathfrak{H})$ obtained by reading the orbits of hyperplanes successively crossed along γ. Verify that, if γ is a path obtained from γ by fliping a 4-cycle or adding or removing a backtrack, then $w(\gamma)$ and $w(\gamma')$ are equal in $C(\mathfrak{H})$. Verify that $w(g\gamma) = w(\gamma)$ for every $g \in G$.

2. Fix a vertex $o \in X$, prove that

$$\Theta : \left\{ \begin{array}{ccc} G & \to & C(\mathfrak{H}) \\ g & \mapsto & w(\text{path from } o \text{ to } g \cdot o) \end{array} \right.$$

defines a morphism.

3. Assume that for every hyperplane J and every element $g \in G$, the hyperplanes J and gJ are neither transverse nor tangent. Also, assume that, for all transverse hyperplanes J_1, J_2 and every element $g \in G$, gJ_2 is not tangent to J_1. Prove that, for every geodesic γ in X, $w(\gamma)$ is an $\mathfrak{H}$-reduced word. Deduce that Θ is then injective.

Exercise 7. Let $\mathcal{P}$ be a semigroup presentation.

1. Prove that, if $\mathcal{P}$ is finite (i.e. has finitely many generators and relations), then the graph $M(\mathcal{P})$ is locally finite.

2. Prove that $M(\mathcal{P}, x)$ has infinite cubical dimension when $\mathcal{P} = \langle x = x^2 \rangle$.

3. Prove that $M(\mathcal{P}, w)$ has finite cubical dimension if there are only finitely many words equal to w modulo $\mathcal{P}$. Show that the converse does not hold.