Canonical Factor of Cellular Automata

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Abstract

We study factor subshifts, column factors and the canonical factor of cellular automata in a large setting, that is as endomorphisms of subshifts. Homogeneity of cellular automata makes them share some dynamical properties with their canonical factor; we review some of them.

Introduction

In symbolic spaces, that is spaces of infinite words, it is common to relate topological notions to language notions, with help of the correspondence between the finite words and the cylinders which form a base of the Tychonoff topology. Many properties of topological dynamics can hence be seen when looking at how some patterns of the configuration evolve.

In the case of one-dimensional cellular automata, the homogeneity of the dynamics is such that a pattern of bounded length is enough, in many cases, to derive a property over patterns of any length (which correspond to open sets of arbitrarily small diameter). Interesting results can be obtained by decomposing the evolution of the cells with respect to the evolution of a finite pattern. We give some of them, which consist in classical facts in the theory of cellular automata, and present here a slight generalization to cellular automata defined over SFTs.

In the first section, we give the main definitions of topological and symbolic dynamics. In the second and third sections, we define the fundamental notion of the article, that is the trace, and give some general facts for dynamical systems and cellular automata. In the last three sections, we deal with the three topological notions that are equicontinuity, expansivity and entropy of systems, and present how they are linked to the notion of trace.

1 Definitions

Dynamical and symbolic systems. A **dynamical system** is (here) a pair (X, F), where X is a compact metric space and $F : X \to X$ a continuous self-map of X. We shall often omit X when it can be understood from the context. A **subsystem** of (X, F) is its restriction (Y, F) to some closed F-invariant subset $Y \subset X$.

A dynamical system induces an action of the monoid $\mathbb{T}$ on X, where $\mathbb{T} = \mathbb{N}$ in general, or $\mathbb{T} = \mathbb{Z}$ if the dynamical system is bijective. In the sequel, we will use $\mathbb{T}$ as being implicitly fixed from the definition of the dynamical system (*i.e.* it can stand for either $\mathbb{N}$ or $\mathbb{Z}$ if we are dealing with a bijective dynamical system, only for $\mathbb{N}$ otherwise).

A **morphism** between two dynamical systems (X, F) and (Y, G) is a continuous map $\Phi : X \to Y$ such that $\Phi F = G\Phi$. If it is surjective, it is called a **factor map**, and (Y, G) is a **factor** of (X, F). If it is

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bijective, it is called a **conjugacy**, and (X,F) and (Y,G) are **conjugate** to one another. If $(X,F)=(Y,G)$, it is called an **endomorphism**, and can be seen itself as a dynamical system. If, besides, it is bijective, it is called an **automorphism**, and can be seen as a bijective dynamical system.

A **symbolic system** is a dynamical system (X,F) where X is totally disconnected. Equivalently, X admits arbitrarily fine clopen partitions (covers of disjoint nonempty closed open sets of any diameter). In that case it is known that, up to homeomorphism, X can be seen as a subset of $A^{\mathbb{N}}$, for some finite alphabet A, endowed with the product topology of the discrete topology. We will restrict our study to symbolic systems of the form (Σ,F), where $\Sigma\subset A^{\mathbb{M}}$ and $\mathbb{M}=\mathbb{Z}$ or $\mathbb{M}=\mathbb{N}$.

A point $x\in A^{\mathbb{M}}$ is called a **configuration**. For $I\subset\mathbb{M}$, we note x_I the restriction of x to I. I can be for example a closed-open interval noted $[\![i,j[\![$ (or another type of interval, noted similarly), in which case we may allow abuse in the indexes (such as assuming that $x_{[\![i,i+k[\![}\in A^k$). Note also that, for $k\in\mathbb{N}$, A^k is the same as $A^{[\![0,k[\![}$. For $k\in\mathbb{N}$, we note $\langle k\rangle=\{\,i\in\mathbb{M}|\,|i|\le k\}$, that is $[\![-k,k]\!]$ or $[\![0,k]\!]$ whether $\mathbb{M}$ is $\mathbb{Z}$ or $\mathbb{N}$.

Shifts. We define the **shift** on $A^{\mathbb{M}}$ as the particular symbolic system σ defined for $x\in A^{\mathbb{M}}$ and $i\in\mathbb{M}$ by $\sigma(x)_i=x_{i+1}$. It is bijective if $\mathbb{M}=\mathbb{Z}$ (hence we will assume $\mathbb{T}=\mathbb{M}$).

A **subshift** is a subsystem (Σ,σ) of the shift $(A^{\mathbb{M}},\sigma)$, *i.e.* its restriction to some closed set $\Sigma\subset A^{\mathbb{M}}$ which is invariant by σ^i for any $i\in\mathbb{M}$. For the sake of simplicity, we shall write that Σ is the subshift. The **language over** $I\subset\mathbb{M}$ of Σ is $\mathcal{L}_I(\Sigma)=\{\,x_I|\,x\in\Sigma\}$. Σ is characterized by its **language** $\mathcal{L}(\Sigma)=\bigcup_{k\in\mathbb{N}}\mathcal{L}_k(\Sigma)$, where $\mathcal{L}_k(\Sigma)=\mathcal{L}_{[\![0,k[\![}(\Sigma)$, consisting of all the finite patterns that appear in some of its configurations. If $k\in\mathbb{N}\setminus\{0\}$ and $\mathcal{F}\subset A^k$ are such that $\Sigma=\left\{x\in A^{\mathbb{M}}\middle|\,\forall i\in\mathbb{M},x_{[\![i,i+k[\![}\notin F\right\}$, then we say that it is an **subshift of finite type** (or SFT) of **order** k.

Fixed a subshift $\Sigma\subset A^{\mathbb{M}}$, the **cylinder** of finite **support** $I\subset\mathbb{M}$ and **central pattern** $u\in A^I$ is the clopen set $[u]=\{\,x\in\Sigma|\,x_I=u\}$. The **central cylinders** $[u]$, for $u\in A^{\langle k\rangle}$ and $k\in\mathbb{N}$, actually form a base for the topology. When $u\in A^k$ for some $k\in\mathbb{N}$, we can note $[u]_i=\sigma^{-i}[u]=\left\{x\in\Sigma\middle|\,x_{[\![i,i+k[\![}=u\right\}$.

Cellular automata. A **cellular automaton** over some surjective subshift $\Sigma\subset A^{\mathbb{M}}$ is an endomorphism of it, *i.e.* a dynamical system which commutes with the shift map.

Hedlund's theorem gives a characterization emphasizing the locality aspect of the map.

Theorem 1 ([1]). *Let $\Sigma\subset A^{\mathbb{M}}$ and $\Gamma\subset B^{\mathbb{M}}$ be two subshifts and Φ a morphism from Σ into Γ. Then there exists a finite **neighborhood** $I\in\mathbb{M}$ and a **local rule** $\phi:\mathcal{L}_I(\Sigma)\to B$ such that for any configuration $x\in\Sigma$ and any cell $i\in\mathbb{M}$, $\Phi(x)_i=\phi(x_{i+I})$.*

Proof. As clopen sets, each of the $|B|$ preimages $\Phi^{-1}([b])$, for $b\in B$, can be decomposed into a finite union of cylinders. Let I be a neighborhood containing all the supports of these cylinders. By construction, $\Phi(x)_0$ depends only on the values of x_I. By shift-invariance, for any $i\in\mathbb{M}$, $\Phi(x)_i=\sigma^i\Phi(x)_0=\Phi\sigma^i(x)_0$ depends only on the values of $\sigma^i(x)_I$. □

In particular, any cellular automaton F over a subshift $\Sigma\subset A^{\mathbb{M}}$ admits an **anchor** $m\in\mathbb{N}$, an **anticipation** $n\in\mathbb{N}$ and a **local rule** $f:\mathcal{L}_{[\![-m,n]\!]}(\Sigma)\to A$ such that for any configuration $x\in\Sigma$ and any cell $i\in\mathbb{Z}$, $F(x)_i=f(x_{[\![-m,n]\!]})$. If $\mathbb{M}=\mathbb{Z}$, we can suppose the anchor and anticipation equal, in which case we call $r=m=n$ **radius** of the cellular automaton. If $\mathbb{M}=\mathbb{N}$, we can suppose the anchor to be 0, in which case we call $r=n$ **radius**.

We will say that Φ is a **letter-to-letter factor map** if we can take a trivial neighborhood $I=\{0\}$.

Inverse limits. The following remark justifies more or less the "factor" terminology.

Remark 1. *If Φ is a factor map of (Y,G) onto (X,F), then (Y,G) is conjugate to the system $(X,F)\otimes_\Phi(Y,G)=(X\otimes_\Phi Y,F\times G)$, where $X\otimes_\Phi Y=\{\,(x,y)\in X\times Y|\,\Phi(y)=x\}$ and $F\times G:(x,y)\mapsto(F(x),G(y))$.*

Furthermore, by a direct induction it can be seen that if $(X_i,F_i)_{0\le i\le l}$ is a finite collection of dynamical systems, $l\in\mathbb{N}$ and $(\Phi_i:X_{i+1}\to X_i)_{0\le i<l}$ a corresponding collection of factor maps, then

$\bigotimes_{(\Phi_i)_{0\leq i<l}}(X_i,F_i) = (X_0,F_0) \otimes_{\Phi_0} \dots \otimes_{\Phi_{l-1}} (X_l,F_l)$ is conjugate to (X_l,F_l). The inverse limit represents some kind of infinite generalization of the $\otimes$ operation.

The **inverse limit** of the sequence $(X_i,F_i)_{i\in\mathbb{N}}$ of dynamical systems with respect to the sequence $(\Phi_i : X_{i+1} \to X_i)_{i\in\mathbb{N}}$ of factor maps is the system $\bigotimes_{(\Phi_i)_{i\in\mathbb{N}}}(X_i,F_i) = (\bigotimes_{(\Phi_i)_{i\in\mathbb{N}}} X_i, \prod_{i\in\mathbb{N}} F_i)$ defined by:

$$\bigotimes_{(\Phi_i)_{i\in\mathbb{N}}} X_i = \left\{ (x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \middle| \forall i \in \mathbb{N}, \Phi_i(x_{i+1}) = x_i \right\}$$

$$\prod_{i\in\mathbb{N}} F_i : (x_i)_{i\in\mathbb{N}} \mapsto (F_i(x_i))_{i\in\mathbb{N}} \ .$$

It basically represents the minimal system which admits all the systems in the sequence as factors, and with the relevant commutations between factor maps.

2 Traces

2.1 Factor subshifts

Definition 1 (Trace)**.** *Let (X,F) be a symbolic system and $\mathcal{P}$ a clopen partition of X. For any point $x \in X$, there exists a unique clopen set $\mathcal{P}(x) \in \mathcal{P}$ that contains x.*

- *The **trace map** of F relatively to partition $\mathcal{P}$ is defined by:*

$$\begin{array}{rccl} T_F^{\mathcal{P}}: & X & \to & \mathcal{P}^{\mathbb{T}} \\ & x & \mapsto & (\mathcal{P}(F^t(x)))_{t\in\mathbb{T}} \ . \end{array}$$

- *The **trace** of F relatively to $\mathcal{P}$ is its image set $\tau_F^{\mathcal{P}} = T_F^{\mathcal{P}}(X)$.*

If we see $\mathcal{P}^{\mathbb{T}}$ as a symbolic space over alphabet $\mathcal{P}$, then the trace $T_F^{\mathcal{P}}$ is continuous since the preimage $(T_F^{\mathcal{P}})^{-1}([u])$ of any cylinder $[u]$ of $\mathcal{P}^{\mathbb{T}}$ of support $J \subset \mathbb{T}$, with $u \in \mathcal{P}^J$, is a finite intersection $\bigcap_{t\in J} F^{-t}(u_t)$ of open sets. Besides, we easily remark that, for any $x \in X$ and any $t \in \mathbb{T}$:

$$T_F^{\mathcal{P}} F^t(x) = (\mathcal{P}(F^{s+t}(x)))_{s\in\mathbb{T}} = \sigma^t T_F^{\mathcal{P}}(x) \ .$$

Hence the trace $\tau_F^{\mathcal{P}}$ (implicitly endowed with the shift map) is a *factor subshift* of F.

Conversely, any factor map Φ of F onto a subshift $(\Sigma \subset A^{\mathbb{T}}, \sigma)$ is, up to letter renaming, the trace map of F relatively to the partition $\mathcal{P} = \left\{ \Phi^{-1}([a]) \middle| a \in A \right\}$. Hence the factor subshifts of a symbolic system are essentially its traces.

Remark 2. *If $\mathcal{P}$ and $\mathcal{P}'$ are two clopen partitions of a space X such that $\mathcal{P}$ is finer than $\mathcal{P}'$, then we have a decomposition $T_F^{\mathcal{P}'} = \Pi T_F^{\mathcal{P}}$, where $\Pi : \tau_F^{\mathcal{P}} \to \tau_F^{\mathcal{P}'}$ is the letter-to-letter factor map of local rule:*

$$\begin{array}{rccl} \pi : & \mathcal{P} & \to & \mathcal{P}' \\ & U & \mapsto & V \text{ where } U \subset V \ . \end{array}$$

Example 3. *Let $\Sigma \subset A^{\mathbb{M}}$ be a subshift and $\mathcal{P}$ a clopen partition of Σ that distinguishes the letters, i.e. at least as fine as the partition $\{ [a] | a \in A \}$ into cylinders of width 1. Remark 2 gives us a letter-to-letter factor map $\Phi : T_\sigma^{\mathcal{P}}(\Sigma) \to \Sigma$ whose local rule maps a trace to the central letter of some corresponding configuration, i.e. $\Phi = (T_\sigma^{\mathcal{P}})^{-1}$ is a conjugacy of the trace $T_\sigma^{\mathcal{P}}(\Sigma)$ onto Σ.*

2.2 Generators

A **generator** of a symbolic system F over X is a countable family of clopen partitions $(\mathcal{P}_i)_{i\in\mathbb{N}}$ such that any factor map Ψ of F onto some subshift $\Gamma \subset B^{\mathbb{T}}$ can be written $\Psi = \Psi' \tau^{\mathcal{P}_i}$, for some $i \in \mathbb{N}$ and some factor map Ψ'. It is a **letter-to-letter generator** if, besides, Ψ' can be taken a letter-to-letter factor map.

This dynamical notion of generator is linked to the topological notion of base, since the traces relatively to some base of partitions form a generator.

Proposition 1. *Any base of clopen partitions is a letter-to-letter generator.*

Proof. Let $(p_i)_{i\in\mathbb{N}}$ a base of clopen partitions and Ψ a factor map onto some subshift. By previous remarks, we can assume that $\Psi = T_F^{\mathcal{P}}$ for some clopen partition $\mathcal{P}$ of X. Let $i \in \mathbb{N}$ be some index of a partition $\mathcal{P}_i$ finer than $\mathcal{P}$. By Remark 2, there is a letter-to-letter factor map $\Phi : \tau_F^{\mathcal{P}_i} \to \tau_F^{\mathcal{P}}$ such that $T_F^{\mathcal{P}} = \Phi T_F^{\mathcal{P}_i}$. □

We can also link the previously-defined notion of generator with that of inverse limit.

Proposition 2. *If F is the inverse limit of a sequence $(\Sigma_i)_{i\in\mathbb{N}}$ of subshifts, then it admits a generator $(\mathcal{P}_i)_{i\in\mathbb{N}}$ whose trace $\tau_F^{\mathcal{P}_i}$ is conjugate to Σ_i for any $i \in \mathbb{N}$.*

Proof. By definition of the product topology, X admits the family $(\mathcal{P}_i)_{i\in\mathbb{N}}$ as a base of partitions, where $\mathcal{P}_i$ is defined as the preimage by the projection $\pi_{[\![0,i]\!]}$ of the partition into cylinders of width $i+1$ of the subshift $\Sigma_0 \otimes \ldots \otimes \Sigma_i$:

$$\mathcal{P}_i = \left\{ \pi_{[\![0,i]\!]}^{-1}([u]) \middle| u \in \mathcal{L}_{i+1}(\Sigma_0 \otimes \ldots \otimes \Sigma_i) \right\} .$$

Since this projection $\pi_{[\![0,i]\!]}$ does not depend on other systems of the base, we can see that the trace $\tau_F^{\mathcal{P}_i}$ is conjugate to the trace $T_\sigma^{\langle i \rangle}(\Sigma_0 \otimes \ldots \otimes \Sigma_i)$, itself conjugate to $\Sigma_0 \otimes \ldots \otimes \Sigma_i$ by Example 3, and hence to Σ_i by Remark 1. □

The converse is true as soon as we order the traces from the finest to the coarsest.

Proposition 3. *If $(\mathcal{P}_i)_{i\in\mathbb{N}}$ is a fineness-increasing sequence of partitions of X, then any symbolic system (X, F) is essentially the inverse limit (Y, G) of the family of traces $(\tau_F^{\mathcal{P}_i})_{i\in\mathbb{N}}$ (relatively to the canonical projections).*

Proof. The morphism:

$$\begin{array}{rccl} \Phi : & X & \to & Y \\ & x & \mapsto & (T_F^{\mathcal{P}_i}(x))_{i\in\mathbb{N}} \end{array}$$

is bijective since $\Phi^{-1}((y_i)_{i\in\mathbb{N}}) = \bigcap_{i\in\mathbb{N}} (T_F^{\mathcal{P}_i})^{-1}(y_i)$ is an intersection of closed sets, that is decreasing if and only if $y \in Y$, and whose diameter converges towards 0. □

Corollary 4. *For any generator $(\mathcal{P}_i)_{i\in\mathbb{N}}$ of a symbolic system (X, F), there is an increasing sequence $(k_i)_{i\in\mathbb{N}}$ of integers such that (X, F) is the inverse limit of $(\tau_F^{\mathcal{P}_{k_i}})_{i\in\mathbb{N}}$.*

Proof. Let $\mathcal{Q}_i$ the partition into balls of radius 2^{-i}. Let us build, by recurrence on $i \in \mathbb{N}$, indexes k_i and factor maps of $\tau_F^{\mathcal{P}_{k_{i+1}}}$ onto $\tau_F^{\mathcal{P}_{k_i}}$. Initially, we can take $k_0 = 0$. Suppose now that $i \in \mathbb{N}$ and k_i are already built. By Proposition 1, there is a decomposition $T_F^{\mathcal{P}_{k_i}} = \Psi_i T_F^{\mathcal{Q}_{r_i}}$, where $r_i \in \mathbb{N}$ and Ψ_i is a letter-to-letter factor map of $\tau_F^{\mathcal{Q}_{r_i}}$ onto $\tau_F^{\mathcal{P}_{k_i}}$. We can suppose without loss of generality that $r_i \geq i$ (otherwise take the maximum with i and compose with a projection). By hypothesis, there exist an index $k_{i+1} \in \mathbb{N}$ and a factor map Ψ'_i of $\tau_F^{\mathcal{P}_{k_{i+1}}}$ onto $\tau_F^{\mathcal{P}_{r_i}}$, such that $T_F^{\mathcal{Q}_{r_i}} = \Psi'_i T_F^{\mathcal{P}_{k_{i+1}}}$. Hence, we have $T_F^{\mathcal{P}_{k_i}} = \Psi_i \Psi'_i T_F^{\mathcal{P}_{k_{i+1}}}$.
Remark that the product $\prod_{i\in\mathbb{N}} \Psi_i$ is a conjugacy of the inverse limit of the sequence $(\tau_F^{\mathcal{Q}_{r_i}})_{i\in\mathbb{N}}$ onto the inverse limit of the sequence $(\tau_F^{\mathcal{P}_{k_i}})_{i\in\mathbb{N}}$, of inverse conjugacy $\sigma \prod_{i\in\mathbb{N}} \Psi'_i$. By Proposition 3, (X, F) is conjugate to the former, hence to the latter. □

2.3 Column factors

If F is a symbolic system over $\Sigma \subset A^{\mathbb{M}}$, a visually intuitive candidate as a generator is the canonical base consisting of the cylinders.

Definition 2 (Column factor)**.**

- *The **column factor** over the finite support $I \subset \mathbb{M}$ is the trace $\tau_F^I = T_F^I(\Sigma)$, where:*

$$\begin{array}{rccl} T_F^I: & \Sigma & \to & (A^I)^{\mathbb{T}} \\ & x & \mapsto & (F^t(x)_I)_{t \in \mathbb{T}} \end{array} .$$

- *The **central column factor** of radius $r \in \mathbb{N}$ is the trace $\tau_F^{\langle r \rangle}$.*

A central column factor corresponds to an observation of the system evolution through some finite window. The central cylinders being a base of partitions, we can see, as implicitly stated in [2], that the family of central column factors is a letter-to-letter generator and the symbolic system is an inverse limit of it; hence we can essentially restrict our study of factor subshifts to column factors.

3 Traces of cellular automata

The particular case of cellular automata allows shift-invariance to link any column factor to some central column factor. Formally, if F is a cellular automaton over some surjective subshift $\Sigma \subset A^{\mathbb{M}}$, then for any cell $i \in \mathbb{M}$ and any finite support $I \subset \mathbb{M}$, $\tau_F^{i+I} = T_F^I(\sigma^i(\Sigma)) = T_F^I(\Sigma) = \tau_F^I$, up to reindexing of the letters in the words.

Canonical factor. The canonical factor was first defined for onesided cellular automata over full shifts in [3]. It corresponds to a trace whose width is equal to the radius of the cellular automaton, *i.e.* the minimal width that cannot be overpassed by information, since if we cut by this width the configuration into a right and a left part, a cell of the left part of the configuration cannot see, in its neighborhood, any of the cells in the right part. Note that our definition of anchor and anticipation, contrary to some other versions, forces the neighborhood of a cell to contain the cell itself.

Definition 3 (Canonical factor)**.** *Let F be a cellular automaton of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$ on some SFT Σ of order $(m+n)$. The **left canonical factor** of F is the trace τ_F^m whose width is the anchor m. Similarly, its **right canonical factor** is τ_F^n. Its **canonical factor** is the widest between the two, i.e. τ_F^r, where $r = \max(m, n)$ is the radius.*

Note that this definition is actually relevant for any cellular automaton over any surjective SFT; if the anchor is more than the order, then it can be increased, and vice-versa.

Overlap.

Definition 4 (Overlap)**.**

- *Let I an interval of $\mathbb{M}$, J an interval of $\mathbb{T}$ and $z = (z^j)_{j \in J}$ a (finite or infinite) word on alphabet A^I, where $z^j = (z_i^j)_{i \in I}$ for any $j \in J$. For any $I' \subset I$, we write the projection $\pi_{I'}(z) = ((z_i^j)_{i \in I'})_{j \in J}$.*

- *Let $k \in \mathbb{N} \setminus \{0\}$, J an interval of $\mathbb{T}$ and $z = (z^j)_{j \in J}, w = (w^j)_{j \in J}$ two (finite or infinite) words on alphabet A^k. If $\pi_{[\![1,k[\![}(z) = \pi_{[\![0,k-1[\![}(w)$, then we say that these words are **overlapping** and we define their **overlap** $z \odot w \in A^{k+1}$ such that for any $i \in [\![0,k]\!], j \in J$, $(z \odot w)_i^j = z_i^j$ if $i < k$ and $(z \odot w)_i^j = w_{k-1}^j$ if $i = k$.*

- *If Σ and Γ are two subsets of $(A^k)^J$, we define their **overlap** $\Sigma \odot \Gamma = \left\{ z \odot w \middle| z \in \Sigma, w \in \Gamma \text{ and } \pi_{[\![1,k[\![}(z) = \pi_{[\![0,k-1[\![}(w) \right\}$.*

- *If $l \in \mathbb{N}$, we define the l-**overlap** of $\Sigma \subset (A^k)^J$ by recurrence: $\Sigma^{[0]} = \Sigma$ and for $l \geq 0$, $\Sigma^{[l+1]} = \Sigma^{[l]} \odot \Sigma^{[l]}$.*
- *We say that $\Sigma \subset (A^k)^J$ is **self-overlapping** if it is equal to the projections $\pi_{[\![i,i+k[\![}(\Sigma^{[1]})$ of its 1-overlap for $0 \leq i \leq 1$.*

The overlap operation is associative, which allows easy manipulation. In particular, we can note that if $l, l' \in \mathbb{N}$ and $\Sigma \subset (A^k)^J$, then $(\Sigma^{[l]})^{[l']} = \Sigma^{[l+l']}$.

Of course, the overlap operation is increasing: if $\Sigma \subset \Gamma$, then for any $l \in \mathbb{N}$, $\Sigma^{[l]} \subset \Gamma^{[l]}$. We can also make the following remark.

Remark 4. *If $k \in \mathbb{N} \setminus \{0\}$, J an interval of $\mathbb{M}$ and $\Sigma \subset (A^k)^J$, then $\Sigma^{[l]}$ is the biggest set $\Gamma \subset (A^{k+l})^J$ such that for any $i \in [\![0, l]\!]$, $\pi_{[\![i,i+k[\![}(\Gamma) \subset \Sigma$, i.e. :*

$$\Sigma^{[l]} = \left\{ u \in (A^{k+l-1})^J \middle| \forall i \in [\![0, l]\!], \pi_{[\![i,i+k[\![}(u) \in \Sigma \right\} .$$

Let us now see that the self-overlapping property is preserved by the overlap operation.

Lemma 1. *If $k \in \mathbb{N} \setminus \{0\}$, J an interval of $\mathbb{T}$ and $\Sigma \subset (A^k)^J$ a self-overlapping set, then its 1-overlap $\Sigma^{[1]}$ is also self-overlapping.*

Proof. Let us show that $\pi_{[\![0,k]\!]}(\Sigma^{[2]}) = \pi_{[\![1,k+1]\!]}(\Sigma^{[2]}) = \Sigma^{[1]}$. By definition, we already have the inclusion. Conversely, let $z \in \Sigma^{[1]}$; let us show that $z \in \pi_{[\![0,k]\!]}(\Sigma^{[2]})$ (the other projection can be obtained symmetrically). Note that, by definition, $\pi_{[\![1,k]\!]}(z)$ is a word of Σ, hence by hypothesis of $\pi_{[\![0,k[\![}(\Sigma^{[1]})$, there exists $z' \in \Sigma^{[1]}$ such that $\pi_{[\![0,k[\![}(z') = \pi_{[\![1,k]\!]}(z)$. Hence $z \odot z' \in (\Sigma^{[1]})^{[1]}$; z is a projection of $\Sigma^{[2]}$. □

Proposition 5. *Let $k \in \mathbb{N} \setminus \{0\}$, J an interval of $\mathbb{T}$ and $l \in \mathbb{N}$. If $\Sigma \subset (A^k)^J$ is self-overlapping, then it is equal to the projections $\pi_{[\![i,i+k[\![}(\Sigma^{[l]})$ of its l-overlap, for $0 \leq i \leq l$.*

Proof. Let us show this property by recurrence on $l \in \mathbb{N}$. The case $l = 0$ is trivial. Let $l \geq 0$, $k \in \mathbb{N} \setminus \{0\}$ and $\Sigma \subset A^k$ which is self-overlapping; let us show that $\Sigma = \pi_{[\![i,i+k[\![}(\Sigma^{[l+1]})$ for $0 \leq i \leq l+1$. By Lemma 1, we know that $\Sigma^{[1]}$ is self-overlapping; thus we can apply the recurrence hypothesis: $\Sigma^{[1]} = \pi_{[\![i,i+k+1[\![}(\Sigma^{[l+1]})$, for $0 \leq i \leq l+1$. But since Σ equals any projection of width k of $\Sigma^{[1]}$, it also equals any projection of $\Sigma^{[l+1]}$. □

For two configurations $x, y \in A^{\mathbb{M}}$ and a cell $i \in \mathbb{M}$, we define the **joint** $x \oplus_i y$ as the configuration z such that $z_k = x_k$ if $k < i$ and $z_k = y_k$ if $k \geq i$. We say that x and y are k**-overlapping** in cell $i \in \mathbb{M}$, where $k \in \mathbb{N}$, if $x_{[\![i,i+k[\![} = y_{[\![i,i+k[\![}$. In that case, we can see that it is also the case of x and $x \oplus_i y$, or yet of $x \oplus_i y$ and y.

Traces and overlapping. The following proposition formalizes the fact that, if we impose some trace of width $m + n$, the evolution of some configuration can be decomposed into a left and a right part in the sense that cells of each side do not see the other one.

Proposition 6. *If (Σ, F) is a cellular automaton of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$ on a SFT of order $(m + n)$, J an interval of $\mathbb{T}$, $i \in \mathbb{M}$ and $x, y \in \Sigma$ two configurations such that $T_F^{[\![i,i+m+n[\![}(x)_J = T_F^{[\![i,i+m+n[\![}(y)_J$. Then for any step $t \in J$, $F^j(x \oplus_i y) = F^j(x) \oplus_i F^j(y)$.*

Proof. We can see by recurrence on $j \in \mathbb{N} \cap J$ that the neighborhood $F^j(x \oplus_i y)_{[\![k-m,k+n[\![}$ of any cell $k \in \mathbb{M}$ corresponds to the neighborhood $F^j(x)_{[\![k-m,k+n[\![}$ if $k < m$, $F^j(y)_{[\![k-m,k+n[\![}$ otherwise, hence the application of the rule is unaltered. If $j \in J \setminus \mathbb{N}$, we can apply the map F^j to what was obtained in the previous point. □

We can also write a variant of the previous proposition that distinguishes left and right.

Proposition 7. *Let (Σ, F) a cellular automaton of anchor $m \in \mathbb{N}$ and anchor $n \in \mathbb{N}$, $k \in \mathbb{N}$, J an interval of $\mathbb{T}$ and x, y two configurations that coincide on some finite segment $x_{[\![0,k[\![} = y_{[\![0,k[\![}$ and on the extremity right and left canonical factors $T_F^{[\![-m,0[\![}(x)_{[\![0,J[\![} = T_F^{[\![-m,0[\![}(y)_J$ and $T_F^{[\![k,k+n[\![}(x)_J = T_F^{[\![k,k+n[\![}(y)_J$. Then the traces corresponding to the whole segment will coincide: $T_F^{[\![-m,k+n[\![}(x)_J = T_F^{[\![-m,k+n[\![}(y)_J$.*

Proof. First, if $x_{[\![0,\infty[\![} = y_{[\![0,\infty[\![}$ and $T_F^{[\![-m,0[\![}(x)_J = T_F^{[\![-m,0[\![}(y)_J$, then a direct recurrence on $j \in \mathbb{N} \cap J$ allows to see that $F^j(x)_{[\![-m,\infty[\![} = F^j(y)_{[\![-m,\infty[\![}$. If $j \in J \setminus \mathbb{N}$, we can apply the map F^j to what was obtained in the previous point. The same can be done on the right side with $T_F^{[\![k,k+n[\![}$. Combining these two points, we get the expected result. □

Proposition 8. *If (Σ, F) is a cellular automaton and $k \in \mathbb{N}$, then τ_F^k is self-overlapping and for any $l \in \mathbb{N}$, $(\tau_F^k)^{[l]} \subset \tau_F^{k+l}$.*

Proof. Let $x \in \Sigma$ be a configuration. Then for any cell $i \in [\![0,l]\!]$, $\pi_{[\![i,i+k[\![} T_F^{k+l}(x) = T_F^{[\![i,i+k[\![}(x)$, hence $T_F^{k+l}(x)$ is in the overlapping $(\tau_F^k)^{[l]}$. Conversely, for any cell $i \in [\![0,l]\!]$, $\pi_{[\![i,i+k[\![}(\tau_F^{k+l}) = \tau_F^k$, hence τ_F^k is overlapping. □

As a result, overlapping of sufficiently wide column factors are still column factors.

Proposition 9. *Let F be a cellular automaton of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$ on some SFT Σ of order $(m+n)$, $l \in \mathbb{N}$ and $k > m+n$. Then $(\tau_F^k)^{[l]} = \tau_F^{k+l}$.*

Proof. From Proposition 8, we just have to show that the overlapping $(\tau_F^k)^{[l]}$ is included in the trace τ_F^{k+l}, which can be done by recurrence on $l \in \mathbb{N}$. The case $l = 0$ is trivial. Now assume that it is true for some $l \in \mathbb{N}$. Let z be a word of $(\tau_F^k)^{[l+1]}$, which is equal by recurrence hypothesis to $(\tau_F^{k+l})^{[1]}$. Then there exist two configurations x, y such that $T_F^{k+l}(x) = \pi_{[\![0,k+l[\![}(z)$ and $T_F^{[\![1,k+l]\!]}(y) = \pi_{[\![1,k+l]\!]}(z)$.
By hypothesis, $k + l - 1 \geq m + n$, hence Proposition 6 gives $T_F^{k+l+1}(x \oplus_1 y) = z$. □

Combining Propositions 9 and 3 allows to rebuild the cellular automaton from its sufficiently wide traces.

Corollary 10. *A cellular automaton F of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$ is essentially the inverse limit of the overlaps $((\tau_F^{m+n+1})^{[k]})_{k \in \mathbb{N}}$ of the trace whose width is the diameter $m+n+1$ of the neighborhood.*

Compatibility.

Definition 5 (Compatible subshift)**.**

- *A word (or configuration) $(z_j)_{j \in J} \subset (A^k)^J$, with $k \in \mathbb{N}$ and $J \subset \mathbb{T}$, is **compatible** with the cellular automaton (Σ, F) of anchor $m \in \mathbb{N}$, anticipation $n \in \mathbb{N}$, and local rule $f : \mathcal{L}_{m+n+1}(\Sigma) \to A$ if for any step $j \in J$, $z^j \in \mathcal{L}_k(\Sigma)$ and if $j+1 \in J$ then for any cell $i \in [\![m, k-n[\![$, $z_i^{j+1} = f(z^j_{[\![i-m,i+n[\![})$.*
- *A subshift $\Sigma \subset (A^k)^{\mathbb{M}}$ is **compatible** with (Σ, F) if all of its configurations are.*

Note that this definition is really relevant when k is at least equal to the diameter of F. In that case, Σ represents a candidate for being the trace of width k, in the sense that all the central cells respect the local rule of F.

Remark 5. *Any column factor τ_F^k of width $k \in \mathbb{N}^*$ of a cellular automaton F is compatible with F.*

Conversely, it can be shown that any compatible self-overlapping subshift is a "column subfactor".

Proposition 11. *If $k \in \mathbb{N}$ and $\Sigma \subset (A^k)^{\mathbb{M}}$ is a subshift which is self-overlapping and compatible with some cellular automaton F over some SFT of order $m+n$, with anchor $m \in \mathbb{N}$, anticipation $n \in \mathbb{N}$, then Σ is included in the trace τ_F^{m+n}.*

Proof. Suppose $\mathbb{M} = \mathbb{Z}$ (the construction is similar when $\mathbb{M} = \mathbb{N}$) and $z \in \Sigma$. Being self-overlapping, we can inductively build a sequence $(z^l)_{l\in\mathbb{N}}$ of words, with $z^0 = z$ and for any $l \in \mathbb{N}$, $z^l \in \Sigma^{[2l]}$ and $z^l = \pi_{[\![1,k+2l]\!]}(z^{l+1})$. For $j \in \mathbb{N}$, we define x^j as the unique element of the decreasing intersection $\bigcap_{l\in\mathbb{N}}[z^l_j]_{-l}$. The compatibility and a direct recurrence give that for any cell $i \in \mathbb{Z}$, $x^{j+1}_i = f(x^j_{[\![i-m,i+n[\![})$, where f is the local rule of F. In particular, we can then see that $\tau^k_F(x^0) = z$. □

The previous result is still true if we take wider m and n, since they can then still be considered as the anchor and anticipation of the same cellular automaton, and their sum can still be considered as the order of the same subshift.

4 Equicontinuity

Let (X, F) be a dynamical system, $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$. A point $x \in X$ is said **ε-unstable** if for any radius $\delta > 0$, there is a point $y \in \mathcal{B}_\delta(x)$ and a step $t \in \mathbb{T}$ for which $\mathbb{d}(F^t(x), F^t(y)) > \varepsilon$. Otherwise the point is said **ε-stable**. A point which is ε-stable for any $\varepsilon > 0$ is said **equicontinuous**.

A dynamical system F is said **ε-sensitive** if all of its points are ε-unstable for some $\varepsilon > 0$. It is said **almost equicontinuous** if its set of equicontinuous points is a residual. It is **equicontinuous** if for any radius $\varepsilon > 0$, there exists a radius $\delta > 0$ such that for all points $x, y \in X$ with $\mathbb{d}(x, y) < \delta$ and all steps $t \in \mathbb{T}$ we have $\mathbb{d}(F^t(x), F^t(y)) < \varepsilon$. Due to the compactness of the underlying space, it is possible to invert the two quantifiers in the definition of equicontinuity: a dynamical system F is equicontinuous if and only if all of its points are. Moreover, in the case of bijective systems, it is known that the definitions with $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \mathbb{N}$ coincide.

Blocking words. The topological notion of equicontinuity can, in a one-dimensional space, be expressed symbolically in terms of blocking words, which prevent the information transmission.

A word $w \in A^*$ is (i, k)-**blocking** (or simply k-**blocking**), with $i \in \mathbb{M}, k \in \mathbb{N}$, for the symbolic system (Σ, F) if $\forall x, y \in [w]_i, \forall t \in \mathbb{T}, F^t(x)_{[\![0,k[\![} = F^t(y)_{[\![0,k[\![}$. Except in trivial cases, we will have $i + k \leq |w|$.

Of course, a word is k-blocking whenever one of its subwords is. Moreover, a k-blocking word is l-blocking for any $l \leq k$.

From the definition, we can see a strong link between word blockingness and point stability.

Remark 6. *Let (Σ, F) be a symbolic system and $k \in \mathbb{N}$. Then the following are true.*

- *A configuration $x \in \Sigma$ is 2^{-k}-stable if and only if $x_{\langle l\rangle}$ is k-blocking for some $l \in \mathbb{N}$.*
- *F is 2^{-k}-sensitive if and only if it admits no k-blocking word.*
- *A configuration is equicontinuous if and only if it admits k-blocking central patterns for any $k \in \mathbb{N}$.*

Blocking words are particularly relevant in cellular automata, in which case a particular blocking width is enough to have any. More formally, if (Σ, F) is a cellular automaton of anchor m and anticipation n, w an (i, m)-blocking (resp. (i, n)-blocking) word and $x \in [w]_i$, then for any configuration $y \in \Sigma$ such that $y_{[\![i,\infty[\![} = x_{[\![i,\infty[\![}$ (resp. $y_{]\!]-\infty,|w|+i[\![} = x_{]\!]-\infty,|w|+i[\![}$) and any step $t \in \mathbb{T}$, we have $F^t(y)_{[\![0,\infty[\![} = F^t(x)_{[\![0,\infty[\![}$ (resp. $F^t(y)_{]\!]-\infty,n[\![} = F^t(x)_{]\!]-\infty,n[\![}$). In other words, the information cannot be transmitted through cells that have known a blocking word as wide as the radius.

Remark 7. *If $(\Sigma \subset A^{\mathbb{M}}, F)$ is a cellular automaton of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$, $i, j \in \mathbb{M}$, $k \geq m$, $l \geq n$, u an (i, k)-blocking word, v a (j, l)-blocking word, and w such that the concatenation uwv is in the language $\mathcal{L}(\Sigma)$, then uwv is $(i, |u| + i + |w| - j + l)$-blocking.*

The previous remark strongly uses the one-dimensional structure of the space to concatenate blocking words, as do the results following from it.

The next result comes essentially from [2].

Proposition 12. *Let (Σ, F) be a cellular automaton of radius r. Then F is equicontinuous if and only if there exists some $l \in \mathbb{N}$ such that any word of $\mathcal{L}_l(\Sigma)$ is r-blocking.*

Proof. If for any l we can find some word of A^l which is not $(\max(m,n))$-blocking, then by the increasing property of blocking words and compactness, we can build a configuration which does not admit any $(\max(m,n))$-blocking central pattern, which thus is not equicontinuous by Remark 6. Conversely, if any word of $u \in \mathcal{L}_l(\Sigma)$ is (i_u, r)-blocking and (j_u, r)-blocking for some $i_u, j_u \in \mathbb{Z}$, then any $q \in \mathbb{N}$ and any word of $w \in \mathcal{L}_{2l+q+\max_{u \in \mathcal{L}_l(\Sigma)} j_u}(\Sigma)$ is, by Remark 7, $(l + i_{w_{[\![0,l[\![}} + \left|w_{[\![l,|w|-l[\![}\right| - j_{w_{[\![|w|-l,|w|[\![}} + r)$-blocking, and in particular, thanks to the increasing property, q-blocking. All widths of blockingness are obtained by all sufficiently large words, and Remark 6 gives that all the configurations are equicontinuous. □

A dynamical system (X, F) is **nonwandering** (resp. **transitive**) if for any nonempty open set $U \subset X$ (resp. and $V \subset X$), there is some point $x \in U$ and some step $t \in \mathbb{T} \setminus \{0\}$ such that $F^t(x) \in U$ (resp. $F^t(x) \in V$). In the case of bijective systems, it is known that the definitions with $\mathbb{T} = \mathbb{Z}$ or $\mathbb{T} = \mathbb{N}$ coincide. Moreover, thanks to the cylinder base, it is easy to see that a subshift Σ is nonwandering (resp. transitive) if and only if for any word $u \in \mathcal{L}(\Sigma)$ (resp. and $v \in \mathcal{L}(\Sigma)$), there exists a word w such that $uwu \in \mathcal{L}(\Sigma)$ (resp. $uwv \in \mathcal{L}(\Sigma)$). Of course it is the case for the full shift, and for many others.

We can actually apply infinitely the previous remark. Let $u \in A^*$ and $U_l = \bigcup_{j>l}[u]_j$ for $l \in \mathbb{N}$. If Σ is nonwandering, then by induction (and thanks to compactness) the intersection $\bigcap_{l\in\mathbb{N}} U_l$ is nonempty. If Σ is transitive, then each U_l is dense and (thanks to Baire's theorem) so is the intersection $\bigcap_{l\in\mathbb{N}} U_l$. If $\mathbb{M} = \mathbb{Z}$, the same can be done with $V_l = \bigcup_{j>l}[u]_{-j}$ for $l \in \mathbb{N}$, and $\bigcap_{l\in\mathbb{N}} U_l \cap V_l$. Applying this to the blocking words, we get the well-known dichotomy between sensitivity and density of equicontinuous points.

Theorem 2. *Let (Σ, F) be a nonsensitive cellular automaton of radius r. If Σ is nonwandering, then F admits some equicontinuous configuration. Moreover, if Σ is transitive, then F is quasiequicontinuous.*

The converse comes directly from the definition: a sensitive system cannot have any equicontinuous point. Theorem 2 gives us the following.

Corollary 13. *If F is a sensitive cellular automaton, then it cannot have any r-blocking words, hence it is 2^r-sensitive.*

Proof of Theorem 2. Suppose that $u \in A^*$ is an (i,k)-blocking word for F, with $k \geq r$, and $U = \bigcap_{l\in\mathbb{N}} \bigcup_{j>l}[u]_j$ if $\mathbb{M} = \mathbb{N}$, $U = \bigcap_{l\in\mathbb{N}} \left(\bigcup_{j>l}[u]_j \cap \bigcup_{j>l}[u]_{-j}\right)$ if $\mathbb{M} = \mathbb{Z}$. By Remarks 7 and 6, any configuration of U is equicontinuous. We have already seen that such a set U is nonempty if Σ is nonwandering, and is dense if Σ is transitive. □

Preperiodicity. We say that a dynamical system (X, F) is **preperiodic** if there exist a period $p \in \mathbb{N} \setminus \{0\}$ and a preperiod $q \in \mathbb{N}$ such that $F^{q+p} = F^q$. In particular, if, besides, F is surjective, then F is periodic. It is also well known that, for subshifts, this condition is equivalent to finiteness.

Remark 8. *The stability of a configuration x of some symbolic system $(\mathit{\Lambda}, F)$ can be expressed in terms of traces: x is 2^{-k}-stable, with $k \in \mathbb{N}^*$, if and only if there exists some radius $l \in \mathbb{N}$ such that $T_F^{\langle k \rangle}([x_{\langle l \rangle}])$ is a singleton. Similarly, F is equicontinuous if and only if for any radius $k \in \mathbb{N}^*$ there exists another radius $l \in \mathbb{N}^*$ such that for any word $u \in A^{\langle l \rangle}$, the trace $T_F^{\langle k \rangle}([u])$ is a singleton.*

Proposition 14. *A symbolic system $(\mathit{\Lambda}, F)$ is equicontinuous if and only if all of its traces are finite.*

Proof. Let F be an equicontinuous symbolic system and $k \in \mathbb{N}$. There exists a radius $l \in \mathbb{N}$ such that for any $u \in A^{\langle l \rangle}$, $T_F^{\langle k \rangle}([u])$ is a singleton. Consequently, $\left|\tau_F^{\langle k \rangle}\right| = \left|\bigcup_{u \in A^{\langle l \rangle}} T_F^{\langle k \rangle}([u])\right| \leq \left|A^{\langle l \rangle}\right|$.

Conversely, if $\tau_F^{\langle k \rangle}$ is finite, then, as a subshift, it is (p,q)-preperiodic, for some $p \in \mathbb{N} \setminus \{0\}$ and $q \in \mathbb{N}$. Any point $x \in \mathit{\Lambda}$ is ε-stable, since any point y of the neighborhood $\bigcap_{t<p+q} F^{-t}(\mathcal{B}_\varepsilon(F^t(x)))$ satisfies $\forall t \in \mathbb{N}, \mathrm{d}(F^t(x), F^t(y)) < \varepsilon$. □

In the case of cellular automata, all the traces have as a projection the trace of width 1. Consequently, the period and preperiod are uniform on all the traces, which slightly generalizes a classical result [2, 4].

Corollary 15. *Any cellular automaton is equicontinuous if and only if it is preperiodic.*

Proof. Let F be an equicontinuous cellular automaton on a subshift $\Sigma \subset A^{\mathbb{M}}$. From Proposition 14, τ_F^1 is preperiodic, *i.e.* there are $p \in \mathbb{N} \setminus \{0\}, q \in \mathbb{N}$ such that for any configuration $x \in \Sigma$, $F^{p+q}(x)_0 = F^q(x)_0$. We conclude by strong shift-invariance.
Conversely, it is well known that any preperiodic dynamical system is equicontinuous. □

Regarding sensitivity, it is transmitted from the system to all of its sufficiently fine traces.

Proposition 16. *Let (Σ, F) a ε-sensitive symbolic system and $\mathcal{P}$ a partition of diameter less than ε. Then $\tau_F^{\mathcal{P}}$ is a sensitive subshift.*

Proof. Let $x \in \Sigma$ and $\delta > 0$. By continuity, there exists $\delta' > 0$ such that for any configuration $y \in \mathcal{B}_{\delta'}(x)$, we have $\mathbb{d}(T_F^{\mathcal{P}}(x), T_F^{\mathcal{P}}(y)) < \delta$. The sensitivity of F gives us some configuration $y \in \mathcal{B}_{\delta'}(x)$ and some step $t \in \mathbb{T}$ such that $\mathbb{d}(F^t(x), F^t(y)) > \varepsilon$. Since $\mathcal{P}$ has a smaller diameter, we get $T_F^{\mathcal{P}}(x)_t \neq T_F^{\mathcal{P}}(y)_t$, *i.e.* $\mathbb{d}(\sigma^t T_F^{\mathcal{P}}(x), \sigma^t T_F^{\mathcal{P}}(y)) = 1$, with $\mathbb{d}(T_F^{\mathcal{P}}(x), T_F^{\mathcal{P}}(y)) < \delta$. □

5 Expansivity

Expansivity represents a very strong instability property: the tiniest difference between two initial points will eventually become big in their evolution. Let (X, F) be a dynamical system and $\varepsilon > 0$. F is **ε-expansive** if for any two points $x \neq y \in X$, there exists some step $t \in \mathbb{T}$ at which $\mathbb{d}(F^t(x), F^t(y)) > \varepsilon$.

It is not difficult to see that, for symbolic systems, this property implies that $T_F^{\mathcal{P}}$ is injective whenever $\mathcal{P}$ is a partition of diameter less than ε. Any of these partitions can be seen as a generator by itself. In other words, the system is conjugate to $\tau_F^{\mathcal{P}}$. Of course, subshifts are expansive, and expansivity is a topological notion, so we get that the expansive systems are essentially the subshifts.

We say that a symbolic system (Σ, F), with $\Sigma = A^{\mathbb{M}}$, is **right-expansive** with **width** $k \in \mathbb{N} \setminus \{0\}$ if for any two configurations $x, y \in \Sigma$ such that $x_{[\![0,\infty[\![} \neq y_{[\![0,\infty[\![}$, there exists a step $t \in \mathbb{T}$ with $F^t(x)_{[\![0,k[\![} \neq F^t(y))_{[\![0,k[\![}$. If $\mathbb{M} = \mathbb{N}$, then this is equivalent to expansivity. Otherwise, we can symmetrically define **left-expansivity**. We say that a cellular automaton is expansive with **width** k if it is right-expansive and left-expansive with width k; this coincides with the definitions over dynamical systems.

Note that if (Σ, F) is right-expansive with width k, then for any $l \geq k$, the trace τ_F^l is conjugate to τ_F^k via the projection $\pi_{[\![0,k[\![}$. Otherwise there would exist x, y with $T_F^k(x) = T_F^k(y)$ but $T_F^l(x) \neq T_F^l(y)$ and thus $x \neq y$. In other particular, if $\mathbb{M} = \mathbb{Z}$, we can see that the family of clopen partitions $(\left\{ [u] \middle| u \in A^{[\![-i,k[\![} \right\})_{i \in \mathbb{N}}$ composed of the cylinders arbitrarily large to the left, but bounded on the right, represents a generator. Of course, the same is true in the left-expansive case.

Proposition 17. *Any right-expansive (resp. left-expansive) cellular automaton of anchor $m \in \mathbb{N}$ and anticipation $n \in \mathbb{N}$ on some surjective SFT of order n (resp. m) is right-expansive with width n (resp. left-expansive with width m).*

Proof. Let (Σ, F) be such a cellular automaton and assume that there are two configurations $x, y \in \Sigma$ and some cell $i \in \mathbb{N}$ such that $x_i \neq y_i$ but $T_F^n(x) = T_F^n(y)$. Let $k \in \mathbb{N}$. Since Σ is a surjective SFT of order n, there exist two configurations $x^k \in \sigma^{-k}(x)$ and $y^k \in \sigma^{-k}(y)$ such that for any cells $i < n$, $x_i^k = y_i^k$. These two configurations are distinct but their traces $T_F^{k+n}\sigma^{-k}(x) = T_F^{k+n}\sigma^{-k}(y)$ are equal thanks to Proposition 6; hence the cellular automaton is not right-expansive. The left side can be proved symmetrically. □

Combining the left and right sides, we can bound the width of expansivity for cellular automata F by its radius r; the trace map T_F^r whose width is the radius is then bijective and F is conjugate to τ_F^r.

Expansivity of cellular automata represents a large domain of open questions, which is, in particular, the base of the justification of Nasu's textile systems [5]. This theory allowed to prove (*cf* [6, 7]) that any expansive cellular automaton over sofic mixing subshifts is conjugate to an SFT (equivalently, τ_F^r is an SFT). In the case when $\mathbb{T} = \mathbb{N}$, it is even conjugate to a full shift (of the form $B^{\mathbb{N}}$, where B is an alphabet).

6 Entropy

The **entropy** of some subshift Σ can be defined in the following way:

$$\mathcal{H}(\Sigma) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_n(\Sigma)|}{n} \ .$$

The **entropy** of some symbolic system (X, F) can be defined in the following way:

$$\mathcal{H}(F) = \lim_{k\to\infty} \mathcal{H}(\tau_F^{\langle k\rangle}) = \lim_{k\to\infty} \lim_{n\to\infty} \frac{\log \left|\mathcal{L}_n(\tau_F^{\langle k\rangle})\right|}{n} \ .$$

If F is a cellular automaton with anchor m and anticipation n, then the overlapping gives us the expression:

$$\mathcal{H}(F) = \lim_{k\to\infty} \mathcal{H}((\tau_F^{m+n+1})^{[k]}) \ .$$

If $k \in \mathbb{N}$, then the trace $\tau_F^{\langle 2k\rangle}$ of width $2k$ is essentially included in $(\tau_F^{\langle k\rangle})^2$; we get the very rough bound: $\mathcal{H}(\tau_F^{\langle 2k\rangle}) \le 2\mathcal{H}(\tau_F^{\langle k\rangle})$. On the other hand, it is well known that entropy does not increase through factor maps: $\mathcal{H}(F) \ge \mathcal{H}(\tau_F^{\langle k\rangle})$, which gives the following fact.

Remark 9. *A cellular automaton has null entropy if and only if each of its traces does.*

The entropy of a cellular automaton can also be bounded by that of its right and left canonical factors.

Proposition 18. *Any cellular automaton (Σ, F) of anchor and anticipation $m, n \in \mathbb{N}$ has an entropy which is less or equal to the sum of that of its left and right canonical factors:*

$$\mathcal{H}(F) \le \mathcal{H}(\tau_F^m) + \mathcal{H}(\tau_F^n) \ .$$

Proof. Let $l \in \mathbb{N}$ and $k \ge m + n$. Then by Proposition 7, there is an injection:

$$\begin{array}{rrcl} \phi: & \mathcal{L}_l(\tau_F^k) & \to & \mathcal{L}_l(\tau_F^m) \times \mathcal{L}_{k-m-n}(\Sigma) \times \mathcal{L}_l(\tau_F^n) \\ & z = (z^j)_{0\le j<l} & \mapsto & (\pi_{[\![0,m[\![}(z), z^0_{[\![m,k-n[\![}, \pi_{[\![k-n,k[\![}(z)) \ , \end{array}$$

which gives the following cardinality inequality:

$$\left|\mathcal{L}_l(\tau_F^k)\right| \le |\mathcal{L}_l(\tau_F^m)|\, |\mathcal{L}_{k-m-n}(\Sigma)|\, |\mathcal{L}_l(\tau_F^n)| \ .$$

Taking the logarithm, and deleting the term which is negligible with respect to l, we get the entropy relatively to any partition into cylinders of width $k \ge m + n$:

$$\mathcal{H}(\tau_F^{[\![0,k[\![}) \le \mathcal{H}(\tau_F^m) + \mathcal{H}(\tau_F^n) \ . \quad \square$$

In particular, for a cellular automaton of radius r, we get the bounds: $\mathcal{H}(\tau_F^r) \le \mathcal{H}(F) \le 2\mathcal{H}(\tau_F^r)$.

In the onesided case, we find again the following known equality.

Corollary 19 ([8])**.** *A CA (Σ, F) of anchor 0 and anticipation $r \in \mathbb{N}$ has an entropy which is equal to that of its canonical factor:*

$$\mathcal{H}(F) = \mathcal{H}(\tau_F^r) \ .$$

The same is true for expansive cellular automata, since they are conjugate to their canonical factor. Moreover, we saw in the previous section that any trace $\tau_F^{[\![0,k[\![}$ of width k greater than the radius r is conjugate to the canonical factor τ_F^r, which gives us that any cellular automata which are expansive in some side has the same entropy than its canonical factor

Proposition 20. *A left-expansive (resp. right-expansive) cellular automaton F of radius $r \in \mathbb{N}$ has entropy $\mathcal{H}(F) = \mathcal{H}(\tau_F^r)$.*

In particular, we find the result of [9]: any permutive cellular automaton (whose local rule acts as a permutation over the first or last cell of the neighborhood) is either onesided or expansive in some side, hence its entropy is equal to that of its canonical factor.

This simplification of the entropy expression in particular cases brings a natural hope of generalization.

Question 1 ([9])**.** *Does there exist a (computable) width $k \in \mathbb{N}$ such that $\mathcal{H}(F) = \mathcal{H}(\tau_F^{\langle k \rangle})$?*

Conclusion

Cellular automata are precisely the dynamical systems which can be defined from one of their factor subshifts: each cellular automaton is the inverse limit of overlaps of some subshift. It shares many properties with it, from the extreme case of expansivity, when it exactly behaves like the subshift, to equicontinuity or preperiodicity, when the subshift satisfies so strong properties that it constrains the whole system.

It is now natural to ask which other properties of the global system can be observed in the dynamics of the subshift itself. For instance, this article did not deal much with transitivity and its variants (mixingness...), which are preserved by factor maps. Hence any trace of a transitive cellular automaton is transitive. But, conversely, if F is a cellular automaton of anchor m and anticipation n, need F be transitive whenever its canonical factor is? Other such questions can be asked: is the entropy of F equal to that of this trace? does it factor onto some other cellular automaton G whenever the two corresponding traces do?

Cellular automata can also be defined over higher-dimensional networks, such as $\mathbb{Z}^2$, or the Cayley graph of any monoid. Everything in the article can be generalized to virtually cyclic monoids or groups, but in more complex contexts, the canonical factor does not play the same role, since it does not disconnect the space of cells. Indeed, the ability to stick together two (or a finite number of) parts of space-time diagrams to get a new one is the crucial point which makes one-dimensional dynamics so particular. It is a base argument in many other results [10, 11], whose generalization remains open in higher-dimensional cases. It is already known that the density and existence of equicontinuous points (Theorem 2) knows a two-dimensional counter-example [12].

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